Canadian Journal of Research

Issued by The National Research Council of Canada

VOL. 15, SEC. A. APRIL, 1937 NUMBER 4

THERMAL STRESS IN LONG CYLINDRICAL SHELLS DUE TO TEMPERATURE VARIATION ROUND THE CIRCUMFERENCE, AND THROUGH THE WALL[1]

By J. N. Goodier[2]

Abstract

The thermal stress in thin-walled cylinders of any cross section has been investigated for internal and external temperatures each varying in any manner round the circumference but not in the axial direction. The thickness also may vary round the circumference.

A method is given for calculating the stress from given temperature distributions, whatever the shape of the cross section. The stress is evaluated for uniform, but different, inside and outside temperatures.

The circular cylinder is treated in detail and the stress found for the general case of circumferential variation. It is shown that the maximum stress will depend only on the temperature distributions and the material, and not on the thickness or diameter of the cylinder.

1. The problems of thermal stress, that is, stress due to non-uniform heating, like the ordinary problems of stress and strain in elastic bodies, fall into two groups. The first group can be satisfactorily treated only by solution of the fundamental differential equations of elasticity for appropriate boundary conditions.* The second group consists of those problems which can, for practical purposes, be adequately treated by the approximate but simple and convenient formulas of the subject usually described as the strength of materials. The well-known Bernoulli-Euler formula for the bending of beams is typical of these. The simplification arises from the circumstance of one or two dimensions of the shapes considered—beams, thin plates, and thin shells—being small.

Some of these simple formulas will be employed to solve the problem described by the title of this paper. Temperatures varying along the axis, but not round the circumference, and temperatures uniform, but different, inside and outside a cylinder, have been dealt with by previous writers (4, 6), but only for the particular case of the circular tube. The thin tube of any shape, with completely general variation of temperature over the surfaces, presents a problem of great analytical difficulty. The differential equations involved are not linear. The same is true of the flat plate with general

[1] *Manuscript received December 30, 1936.*
Contribution from the Department of Engineering and Metallurgy, Ontario Research Foundation, Toronto, Canada.

[2] *Research Fellow in Applied Mechanics, Ontario Research Foundation.*

* *This group is treated in a forthcoming paper "On the integration of the thermo-elastic equations."*

surface temperatures. The restricted problem we are concerned with, however, is not only important in engineering, notably in its application to boiler tubes, but also tractable mathematically.

We begin with the incomplete cylinder, *i.e.*, the section $AQPFC$, Fig. 1, is not a closed curve. The results obtained are necessary for the complete cylinder, which, for the present, is supposed converted into an incomplete one by a cut along a generator, equivalent to a gap of zero width. The

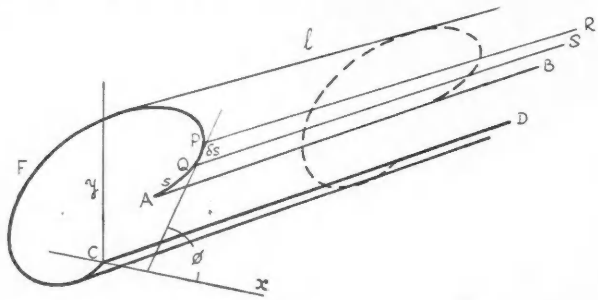

Fig. 1.

problem is reduced, by a physical argument, to the solution of a set of equations. A narrow strip $PQSR$ is considered in isolation. It is given arbitrary inside and outside temperatures, T_i and T_e respectively (the transition through the thickness being linear), while the remainder of the tube is kept unheated. The strip expands, bends and curls, and no longer fits the place from which it was taken. But by the imposition of suitable terminal forces and couples it is so far restored to its original state that it can be supposed reattached to the rest of the tube, so that points originally contiguous are so again without any necessity for straining the rest of the tube. The reassembly, however, involves relative rigid body displacements of the two cold parts PFC and QA, and hence an alteration of the gap AC. This alteration is calculated. Then the stress system required to re-close the gap AC, if the tube be a closed one, is found, and the effects of removing the imposed terminal forces and couples of the strip are considered. The effects of heating over more than an elementary strip follow by integration.

The Incomplete Cylinder

2. For the moment, let the strip $PQSR$ (Figs. 1 and 2) of (the complete) axial length l and infinitesimal arc length δs, have the inside temperature T_i, and outside temperature T_e, the rest of the cylinder being kept at temperature zero.

Let this strip be freed from the rest of the cylinder by cuts along PR and QS. Then it is free to expand, without thermal stress.* It increases its linear dimensions, and also changes its curvature on account of the tem-

* *There is a general theorem that temperature distributions linear in cartesian co-ordinates give rise to no stress (8, p. 204).*

perature gradient through the thickness. If the mean temperature $T_m = \frac{1}{2}(T_i + T_e)$, then the length increases by a uniform linear strain αT_m, *i.e.*, by $l\alpha T_m$, α being the coefficient of linear expansion. The arc length δs increases by $\delta s \,.\, \alpha T_m$. The thickness h increases by $h\alpha T_m$. The length l, which originally had no curvature, becomes curved, as sketched in Fig. 2 (with $T_e > T_i$), to a radius r, where $1/r = (T_e - T_i)\alpha/h$. The arc δs changes its curvature by the same amount.

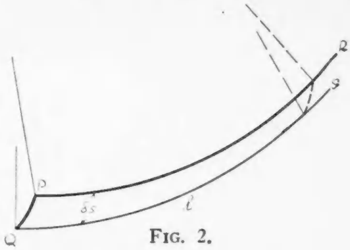

FIG. 2.

3. A compressive stress $E\alpha T_m$, applied to the ends of the strip will exactly cancel the thermal strain αT_m in the direction l (where E is Young's modulus). It will increase δs by a strain $\sigma\alpha T_m$ (where σ is Poisson's ratio) so that the total increase of δs becomes $\delta s(1 + \sigma)\alpha T_m$. It will also increase the thickness h by $h\sigma\alpha T_m$.

A bending moment of suitable magnitude, applied to the ends, will exactly cancel the thermal curvature of l, $(T_e - T_i)\alpha/h$. It is easily found from simple beam formulas that the necessary bending moment is $\frac{1}{12}E\alpha h^2(T_e - T_i)\,\delta s$. The vector representing this moment is in the direction of δs.

This moment will produce an *anticlastic* curvature of δs, of amount $\sigma(T_e - T_i)\alpha/h$, and the total change of curvature of δs is therefore now $(1 + \sigma)(T_e - T_i)\alpha/h$.

The edges PR and QS of the strip are at present thicker than the thickness h of the cold tube by $h(1 + \sigma)\alpha T_m$. In order to bring the faces of the cuts of the strip and of the rest of the tube to the same configuration, it must be supposed that shearing and normal tractions as indicated in Fig. 3, and perhaps others, act on the strip, and equal and opposite tractions on the rest of the tube.

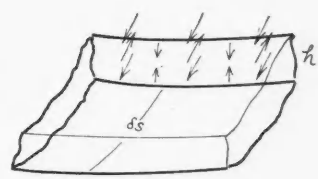

FIG. 3.

It is however neither practicable nor necessary to compute these tractions and their effects. The strip, at present, is regarded as hot, and the rest of the tube as cold. But later the effects of heating all such strips will be added so as to arrive at a tube heated all round, the temperature varying smoothly. Evidently the adjustment of the thickness of one strip to that of its neighbors involves only the temperature differentials. In the limit, therefore, the required tractions on the cuts such as PR and QS will vanish. We may say that they disappear in the process of integration.*

* *The validity of this argument may be better seen by considering a problem of plane stress. Let a long strip be compressed on finite opposite segments on the two edges. The part between these segments may be considered in isolation, and we may inquire what tractions are necessary to fit it back into the strip. We should find that an adjustment of thickness is required, just as in the present problem. But if the compressive forces have a smooth distribution over a succession of infinitesimal segments, the process of integration involved eliminates the tractions that have to be introduced to adjust the thickness, and the stress is plane. This is confirmed by the exact analytical solution of the plane stress problem.*

4. For the present, suppose that the tractions act, so that the thicknesses become equal. Any other possible effects due to the tractions on the strip need not be considered. Then we may suppose that the strip is rejoined to the other two parts of the cylinder. But on account of the alterations in the length and curvature of δs, these parts will no longer have their former relative positions.

Consider any cross section of the cylinder and draw rectangular axes Cx, Cy with origin on CD. In Fig. 1 these are drawn on the end $AQPFC$, for convenience.

Then the displacement of AQ relative to CFP has x and y components given by,

$$-(1 + \sigma)\alpha T_m \cos\phi \,.\, \delta s - (1 + \sigma)\alpha(T_e - T_i)\frac{y}{h}\,\delta s$$

and

$$-(1 + \sigma)\alpha T_m \sin\phi \,.\, \delta s + (1 + \sigma)\alpha(T_e - T_i)\frac{x}{h}\,\delta s \;.$$

The first terms arise from the extension of δs, the second terms from the change in its curvature; the rotation (anticlockwise) of AQ relative to CFP is $-(1 + \sigma)\alpha(T_e - T_i)h^{-1}\delta s$; x and y are the co-ordinates of δs, and ϕ is the angle δs makes with the x-axis.

5. If now all the other strips into which the cylinder may be divided are treated in the same way, each being given its proper temperatures, the displacements δ_x and δ_y and rotation ω of the edge AB relative to the edge CD may be obtained by superposition. Then

$$\left.\begin{aligned}
\delta_x &= -(1 + \sigma)\alpha\int T_m \cos\phi \,.\, ds - (1 + \sigma)\alpha\int (T_e - T_i)\frac{y}{h}\,ds\,,\\
\delta_y &= -(1 + \sigma)\alpha\int T_m \sin\phi \,.\, ds + (1 + \sigma)\alpha\int (T_e - T_i)\frac{x}{h}\,ds\,,\\
\omega &= -(1 + \sigma)\alpha\int (T_e - T_i)\frac{ds}{h}\,,
\end{aligned}\right\} \quad (1)$$

T_m, T_e and T_i being now given functions of s, the arc length. The integrals are taken all round the section.

These displacements and rotation are maintained by the temperature distribution together with:

(i) Stress $-E\alpha T_m$ on the ends of the cylinder;

(ii) A distribution of moment on the ends,

$$m = \tfrac{1}{12}E\alpha h^2(T_e - T_i),$$

per unit length of arc, the vector of m being directed along δs or along the tangent at any point of the end section.

The state of stress anywhere in the wall of the cylinder is given by the same stress and moment. There is no stress other than the axial component.

6. The end stresses (i) and (ii) of Art. 5 may be eliminated by superposing the ordinary isothermal stress distribution having equal and opposite end stress and moment. The complete determination of this distribution is a problem in the theory of thin shells. However, if the cylinder is long, the distribution in the middle part will depend, according to the Principle of Saint-Venant, only on the *resultant* force and couple on the ends. The resultant force is $E\alpha \int hT_m ds$ and the resultant couple has components

$$\left.\begin{aligned} M_x &= -\tfrac{1}{12}E\alpha \int h^2(T_e - T_i)\cos\phi \,.\, ds, \\ M_y &= -\tfrac{1}{12}E\alpha \int h^2(T_e - T_i)\sin\phi \,.\, ds, \end{aligned}\right\} \qquad (2)$$

and from these the corresponding stress in the middle part can be found from the simple beam theory.

The Complete Cylinder

7. If a cut is made along a generator, the formulas (1) give the displacement and rotation of one face of the cut relative to the other, when the end actions (i) and (ii) of Art. 5 are applied, and the cylinder is heated. The integrals are now taken around the circumference.

The faces of the cut may be brought together by a suitable force and couple on one face, an equal and opposite force and couple on the other face, and rejoined. This introduces an additional stress distribution to be superposed on that of Art. 5.

To evaluate it, consider the forces X_o and Y_o and moment M_o, per unit axial length, distributed uniformly along the faces of the cut, acting on the cut tube (Fig. 4). The displacements and rotation of the face A relative to the face C are conveniently found by Castigliano's theorem (7, p. 434). Since we are dealing with a cylinder and not a ring, we use the plate modulus* $E/(1 - \sigma^2)$ instead of Young's modulus, E, and write D for $Eh^3/12(1 - \sigma^2)$. The thickness h will now be taken as constant, although there would be no difficulty in retaining it as a variable. Then,

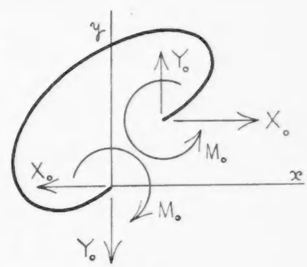

FIG. 4.

$$\left.\begin{aligned} D\delta_x &= M_o\int y ds + X_o\int y^2 ds - Y_o\int xy ds \\ D\delta_y &= -M_o\int x ds - X_o\int xy ds + Y_o\int x^2 ds \\ D\omega &= M_o\int ds + X_o\int y ds - Y_o\int x ds \end{aligned}\right\} \qquad (3)$$

The integrals can be expressed in terms of the mean circumference of the section, L, the co-ordinates of the centroid $\bar{x}$, $\bar{y}$, the moments of inertia I_{xx}, I_{yy}, and the product of inertia I_{xy}, of the area of the section, since $L = \int ds$, $L\bar{x} = \int x ds$, $L\bar{y} = \int y ds$, and $I_{xx} = h\int y^2 ds$, $I_{yy} = h\int x^2 ds$, $I_{xy} = h\int xy ds$. The displacements and rotation given by Equations (3) must be equal and opposite

* *Expressing the effect of preventing anticlastic curvature.*

to the displacements and rotation given by Equations (1). The forces X_o and Y_o and the couple M_o can be found by solving the equations,

$$\left.\begin{aligned} M_oLh\bar{y}+X_oI_{xx}-Y_oI_{xy} &= D(1+\sigma)\alpha[h\textstyle\int T_m \cos\phi\,.\,ds+\int(T_e-T_i)yds]\,, \\ -M_oLh\bar{x}-X_oI_{xy}+Y_oI_{yy} &= D(1+\sigma)\alpha[h\textstyle\int T_m \sin\phi\,.\,ds-\int(T_e-T_i)xds]\,, \\ (M_o+X_o\bar{y}-Y_o\bar{x})Lh &= D(1+\sigma)\alpha\textstyle\int(T_e-T_i)ds\,. \end{aligned}\right\}(4)$$

8. The axial stress is made up of four contributions:

(i) A stress $-E\alpha T_m$ (Art. 3);

(ii) A stress corresponding to the distribution of an axial bending moment $\frac{1}{12}E\alpha h^2(T_e - T_i)$ per unit arc length on the mean circumference (Art. 3);

(iii) The axial stress due to the distribution of traction $E\alpha T_m$ on the *ends*. This cancels, on the ends only, the stress (i);

(iv) The axial stress due to the distribution of the couple on the *ends* $-\frac{1}{12}E\alpha h^2(T_e - T_i)$ per unit arc length on the mean circumference. This cancels, on the ends only, the stress (iii).

The ends are then completely free.

The resultant moments of (iv) are given by Equations (2), and the stress due to them, in the middle part, *i.e.*, away from the ends, can be calculated by the simple beam formulas. The appropriate moments of inertia are not I_{xx} and I_{yy} as defined previously, since the latter are referred to axes not through the centroid of the section. Writing I'_{xx}, I'_{yy} for the centroidal moments of inertia, and ζ for the distance normal to the middle surface measured inwards, the stresses (i) to (iv) may be combined into the complete expression for the axial stress

$$E\alpha\left[-T_m+\frac{\zeta}{h}(T_e-T_i)+\frac{1}{L}\int T_m ds \right.$$
$$\left. +\frac{h^2(x-\bar{x})}{12I'_{xx}}\int(T_e-T_i)\sin\phi\,.\,ds-\frac{h^2(y-\bar{y})}{12I'_{yy}}\int(T_e-T_i)\cos\phi\,.\,ds\right] \quad (5)$$

When the shape of the cylinder, and the temperature distributions, are given, it is a straightforward process to evaluate the coefficients in Equations (4), analytically or graphically, and to solve for X_o, Y_o and M_o, and from these compute the corresponding stress. The axial stress can be found directly from Expression (5).

Simple formulas can be derived for circular, elliptical, rectangular or other regular sections. The further development, with the exception of the next paragraph, will be confined to the circular case.

Cylinder of Any Shape, but of Uniform Thickness, with Uniform but Unequal Inside and Outside Temperatures

9. It is evident that since the temperature difference is constant around the cylinder, the curvature changes due to it in conjunction with the auxiliary end forces and couples will also be constant around the cylinder. Such a

curvature change can be annulled by a moment distribution M_o. Clearly this is what is required to close the gap. It is easily found that

$$M_o = \frac{\alpha}{h} D(1 + \sigma)(T_e - T_i) .$$

The extreme circumferential stress $(6M_o/h^2)$, then, has the values*

$$\pm \frac{E\alpha(T_e - T_i)}{2(1 - \sigma)} .$$

The complete axial stress is given by (ii) of Art 8, and has the extreme values

$$\pm \tfrac{1}{2} E\alpha(T_e - T_i)$$

except near the ends, which, of course, are free of traction. It is noteworthy that these values are independent of the size, shape and thickness of the cylinder, depending only on the material and the temperatures occurring within it.

It is known that these formulas hold for the special case of the circular tube (8, p. 373).

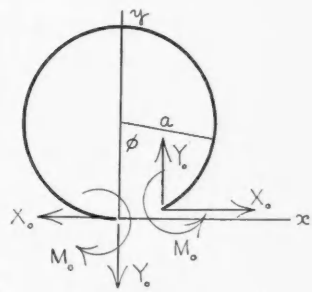

FIG. 5.

The Circular Cylinder

10. When the cylinder has a mean radius a, the expressions on the left of the Equations (4) reduce to $\pi a^2 h(2M_o + 3aX_o)$, $\pi a^3 h Y_o$, $2\pi a(M_o + aX_o)$ respectively, with axes as in Fig. 5; in the integrals on the right, ds is replaced by $a\,d\phi$, x by $a \sin \phi$, and y by $a(1 - \cos \phi)$. The solution of the equations is

$$\left.\begin{aligned}
X_o &= \frac{D}{\pi a^2}(1 + \sigma)\alpha\left[\int_0^{2\pi} T_m \cos \phi\, d\phi - \frac{a}{h}\int_0^{2\pi}(T_e - T_i) \cos \phi\, d\phi\right], \\
Y_o &= \frac{D}{\pi a^2}(1 + \sigma)\alpha\left[\int_0^{2\pi} T_m \sin \phi\, d\phi - \frac{a}{h}\int_0^{2\pi}(T_e - T_i) \sin \phi\, d\phi\right], \\
M_o &= \frac{D}{\pi a}(1 + \sigma)\alpha\left[-\int_0^{2\pi} T_m \cos \phi\, d\phi + \frac{a}{2h}\int_0^{2\pi}(T_e - T_i)(1 + 2\cos \phi)\, d\phi\right].
\end{aligned}\right\} \quad (6)$$

These formulas will give the bending moment, $M_o + X_o y - Y_o x$, per unit axial length, explicitly in terms of the given temperature distributions. The direct stress due to X_o and Y_o will be small, of the order h/a, in comparison with the bending stress.

When the temperature difference $T_e - T_i$ is as great as or greater than the mean temperature $T_m = \frac{1}{2}(T_e + T_i)$, the contributions from the latter will be negligible, being of the order h/a, compared with the contributions arising from the temperature difference.

* *Positive signs pertain to the inside.*

11. The integrals occurring in Expression (6) may be replaced by coefficients in the Fourier expansion of T_e and T_i. Let

$$
\begin{aligned}
T_e &= A_0 + A_1 \cos\phi + A_2 \cos 2\phi + \cdots \\
&\qquad + B_1 \sin\phi + B_2 \sin 2\phi + \cdots \\
T_i &= A_0' + A_1' \cos\phi + A_2' \cos 2\phi + \cdots \\
&\qquad + B_1' \sin\phi + B_2' \sin 2\phi + \cdots
\end{aligned}
$$

Then

$$
2\pi A_0 = \int_0^{2\pi} T_e d\phi \,, \quad \pi A_1 = \int_0^{2\pi} T_e \cos\phi d\phi \,, \quad \pi B_1 = \int_0^{2\pi} T_e \sin\phi d\phi
$$

and A_0', A_1', B_1' are similarly expressed in terms of T_i.

Thus X_o, Y_o and M_o, and the stresses corresponding to them, are independent of the terms in cos 2ϕ, sin 2ϕ, and all higher harmonics of the temperatures.

Replacing the factors $\frac{a}{h} \pm \frac{1}{2}$ by $\frac{a}{h}$, Equations (6) can be written,

$$
\left.
\begin{aligned}
X_o &= -\frac{D}{ah}(1+\sigma)\alpha(A_1 - A_1') \,, \\
Y_o &= -\frac{D}{ah}(1+\sigma)\alpha(B_1 - B_1) \,, \\
M_o &= \frac{D}{h}(1+\sigma)\alpha[A_1 - A_1' + A_0 - A_0'] \,.
\end{aligned}
\right\} \qquad (7)
$$

To obtain the corresponding extreme fibre stress, neglecting the direct stress due to X_o and Y_o, the moment $M = M_o + X_o y - Y_o x$ is multiplied by $6/h^2$. The result is

$$
\pm \frac{E\alpha}{2(1-\sigma)} \left\{ A_0 - A_0' + (A_1 - A_1') \cos\phi + (B_1 - B_1') \sin\phi \right\} \qquad (8)
$$

whence the maximum follows easily. This is independent of the radius and thickness of the cylinder.

The complete expression for the axial stress at the middle part of the length is

$$
\begin{aligned}
E\alpha \Big\{ -T_m + \frac{\zeta}{h}(T_e - T_i) + \tfrac{1}{2}(A_0 + A_0') \\
+ (A_1 - A_1') \frac{h}{12a} \cos\phi + (B_1 - B_1') \frac{h}{12a} \sin\phi \Big\} \,. \qquad (9)
\end{aligned}
$$

The last two terms will be relatively small. When they are neglected this stress becomes independent of a and h.

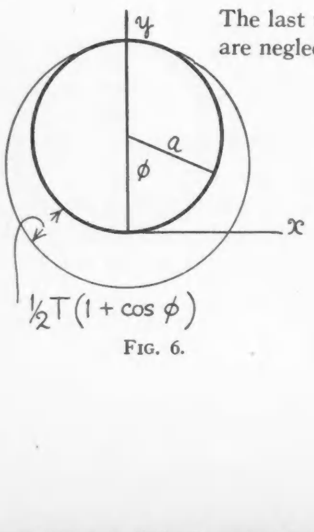

FIG. 6.

12. As an example of thermal stress in a circular cylinder, let the internal temperature be uniform, and let the external temperature exceed the internal by $\frac{1}{2}T(1 + \cos\phi)$. Then T is the maximum temperature difference, and the distribution of temperature difference round the circumference is as indicated by the polar diagram, Fig. 6. It corresponds roughly to a tube heated from below, and containing a well mixed fluid.

A_0 and A_1 are each equal to $\frac{1}{2}T$, and all other Fourier coefficients are zero. Then

$$X_o = -\frac{D}{ah}(1+\sigma)\alpha\frac{T}{2}, \quad Y_o = 0, \text{ and } M_o = \frac{D}{h}(1+\sigma)\alpha T .$$

The moment M is given by

$$\frac{D}{h}(1+\sigma)\alpha T\left(1-\frac{y}{2a}\right).$$

The maximum is at $y = 0$, and corresponds to an extreme fibre stress

$$\pm E\alpha T/2(1-\sigma).$$

To obtain a numerical result, take the internal temperature as 100° C., the maximum external temperature as 200° C., E as 3×10^7 lb. per sq. in., α as 11.6×10^{-6} per °C., σ as 0.3 (as for steel). Then $E\alpha T/2(1-\sigma)$ is nearly equal to 25,000 lb. per sq. in.

The axial stress can be found from Expression (9). With the assumed distribution, there is a compressive stress in the hottest part ($\phi = y = o$) nearly equal to $\frac{3}{4}E\alpha T$, or 27,000 lb. per sq. in. for the values taken. This is the value for the stress at the inside. At the outside, the axial stress is 9,000 lb. per sq. in. tension.

13. As a final example, let the outside of the cylinder be heated to a temperature T, kept constant over a small arc of circumferential length λ, and let the rest of the outside surface* and the whole inside surface be kept at zero temperature. This is a rough representation of a distribution that might occur in a welding process.

It is easily shown that the maximum circumferential stress is given by

$$\frac{3}{4}\frac{E\alpha T}{1-\sigma}\cdot\frac{\lambda}{a}.$$

Since this is proportional to λ/a, it shows that localized heating does not produce large circumferential stresses. This is true, of course, only while there is no *axial* variation of temperature.

As to the axial stress under such conditions, the parts (i) and (ii) of Art. 8 are confined to the heated strip and occur throughout its length: (iii) and (iv) are distributions due to forces and moments localized on the ends of the heated strip. In the interior these will set up only very small stresses. In the interior, therefore, there will be an axial stress, given by $-\frac{1}{2}E\alpha T(1 \pm 2\zeta/h)$, confined to the heated strip, or a compressive stress varying from zero at the inside to $E\alpha T$ at the outside.

The stress at the ends of the cylinder presents a separate problem, and can be dealt with approximately by methods given in another paper (2). It may

* *This disregard of the restriction to smooth temperature variations, referred to in Art. 3, is of no more importance here than its equivalent in plane stress systems with discontinuous boundary stress. The discontinuity conveniently replaces a variation as abrupt as is allowed, without seriously affecting the conclusions.*

be shown that there would be local tensile circumferential stresses at the ends with a maximum value $2E\alpha T/(3 + \sigma)$.

14. The thermal stress in a *thick* cylinder with a steady-state temperature distribution due to surface temperatures arbitrarily distributed round the inside and outside surfaces, but independent of the axial co-ordinate, can be reduced, as was shown by N. Muschelišvili (5) and M. A. Biot (1), to an axial stress, and a dislocational stress corresponding to the stress specified above by means of X_o, Y_o and M_o. In order to find the appropriate magnitudes of the dislocations it is necessary first to obtain the solution of the potential problem of finding the temperature distribution from the surface temperatures.

By restricting the investigation described in this paper to the thin cylinder, the potential problem is avoided, and it becomes possible to find the stress and strain in terms of arbitrarily given surface temperatures for a cylinder of any shape and variable thickness. It should be said, however, that the potential problem presents no difficulties in the particular case of the uniform, thick circular cylinder. Explicit formulas for this case are given in another paper (3).

References

1. Biot, M. A. Phil. Mag. 19 : 540-549. 1935.
2. Goodier, J. N. Physics, 7 : 156-159. 1936.
3. Goodier, J. N. J. Applied Mechanics, March. 1937.
4. Kent, C. H. Trans. Am. Soc. Engrs. 53 : 167-180. 1931.
5. Muschelišvili, N. Bulletin de l'Université de Tiflis, 3. 1923.
6. Timoshenko, S. and Lessells, J. M. Applied elasticity. East Pittsburgh, Pa., Westinghouse Technical Night School Press. 1925.
7. Timoshenko, S. Strength of materials. Vol. II. D. Van Nostrand Company, Inc. New York. 1930.
8. Timoshenko, S. Theory of elasticity. McGraw-Hill Book Company. New York. 1934.

THE SPECIFIC HEAT OF COPPER FROM 30° TO 200° K.[1]

By S. M. Dockerty[2]

Abstract

An all-metal adiabatic vacuum calorimeter was used to determine the specific heat of copper over small temperature intervals from 30° to 200° K. The accuracy is considered to be within 0.05% for the greater part of the range. A curve is given showing the variation with temperature of the characteristic temperature, θ, for copper.

Introduction

Bronson, Chisholm, and Dockerty (2, 4) described the development of an all-metal adiabatic calorimeter and its use in precision measurements of the specific heat of copper from −80° to 100° C. The present paper deals with the construction of a similar type of calorimeter for much lower temperatures, used to determine the specific heat of copper from 30° to 200° K.

The method, that of electrical heating under adiabatic conditions, has also been described (2, 4). The general features can be seen from the diagram of the apparatus in Fig. 1. The calorimeter is cooled down to the lowest

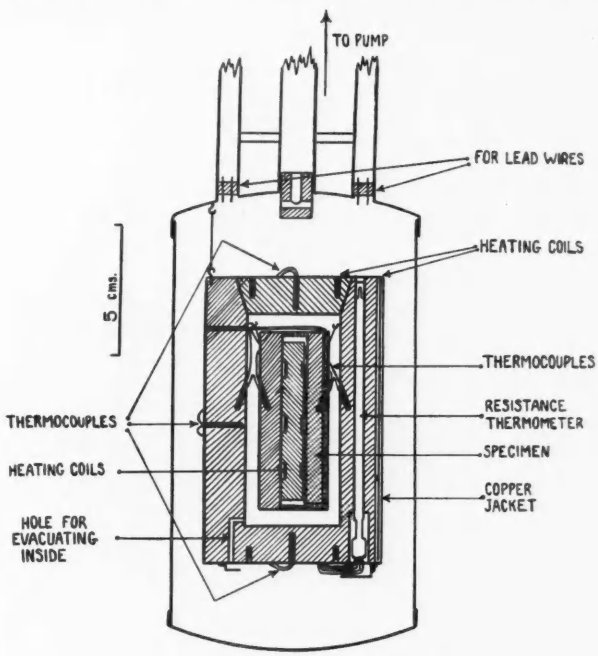

Fig. 1. *Adiabatic calorimeter.*

[1] *Manuscript received March 5, 1937.*
Contribution from the Department of Physics, University of Toronto, Toronto, Canada.
[2] *Holder of a bursary (1932–1933), studentship (1933–1934) and fellowship (1934–1935) under the National Research Council of Canada; Graduate student of the University of Toronto (1933–1936).*

point of the range at which one wishes to work, by immersing the container in a low temperature bath, the rate of cooling being speeded up somewhat by having hydrogen gas in the container.

When the desired temperature is reached the container is evacuated. The specimen is heated so as to change its temperature in steps of 5° to 8°, the times of heating varying from 15 to 25 min. Adiabatic conditions are obtained by keeping the temperature of the jacket as nearly as possible equal to that of the specimen, as indicated by the differential thermocouples. The temperatures are so adjusted, before and after the heating period, that the differences between the jacket and the specimen are less than 0.001° C. Thus the temperature of the jacket, as measured by the resistance thermometers, will be also the temperature of the specimen. The electrical energy supplied was measured, to within 0.02%, with a potentiometer and a standard cell that was recently calibrated in the laboratories of the National Research Council of Canada. The mass of the specimen was known to within 0.01%. Temperature differences were measured to within 0.02% for the greater part of the range, the precision being somewhat less than this at low temperatures.

Apparatus

Jacket

Some difficulty was experienced with temperature gradients in the jacket of the older apparatus owing to the faulty distribution of heat loss. This was largely avoided in the present apparatus by using pure copper for the jacket, and by high evacuation which increased the thermal resistance between the jacket and the container about five times. Temperature gradients that were due to non-uniform heating were eliminated as before by winding separate heating coils on the sides and ends of the jacket; because of the high vacuum it was necessary to cement these coils in deep grooves.

Differential thermocouples between the sides and ends of the jacket, as shown in Fig. 1, served to indicate temperature inequalities over the inner surface. These were put in from the outside and were left in permanently, so that adjustments could be made during the actual experiments.

Two jackets were made by machining a 3-in. bar of copper. The first, which was used for most of the measurements above liquid air temperature, was about one and one-half times as large as that shown in Fig. 1, and differed from it in that it was divided horizontally at the centre into two symmetrical sections. This feature proved somewhat troublesome because it was found difficult to keep both sections at the same temperature. This equality was necessary because the resistance thermometers would indicate the temperature of the section in which their coils were situated. In addition, it was found that, because of the high vacuum, the size could be reduced considerably without affecting the accuracy of the results; this feature was desirable for the cooling to liquid hydrogen temperatures. A new jacket was therefore made as shown in Fig. 1. Care was taken to keep the end plug at the same temperature as the rest of the jacket, but a slight difference

would have no serious effect. Small leads of copper wire were used to prevent the production of cold spots on the jackets at low temperatures where the thermal conductivity of copper becomes quite high.

Copper Specimen

This consisted of a bar of commercially pure cold-rolled copper, a copper plug, and an 80-ohm manganin heating coil fitted together as shown in Fig. 1. The weights of the specimens used with the large and small jackets were about 650 and 320 gm. respectively. The amount of foreign material such as silk insulation, silk suspension, cement, and shellac was reduced to less than 0.1% of the total copper equivalent.

Because of the high vacuum it was necessary to bind the heating coil to the plug with a thin coat of shellac. Small copper leads were used, and care was taken that these should be in good thermal contact with both the specimen and the jacket.

The two-junction copper-constantan thermocouples were cemented in small holes in the specimen and the jacket respectively as shown. With a sensitive galvanometer these thermocouples would indicate temperature differences between the specimen and the jacket of less than 0.001° C.

Resistance Thermometers

Temperatures were measured with metal-stem resistance thermometers of platinum and lead placed in holes bored lengthwise in the jacket. To prevent heating and lagging effects it was necessary that the thermometers should not be evacuated with the rest of the system. They were therefore filled with helium gas at a pressure of one atmosphere, and sealed. A very satisfactory type of seal, which holds at low temperatures, is shown in Fig. 2.

Resistances were measured with a carefully calibrated Müller bridge maintained at a constant temperature in a thermostatic box. Small copper wire leads were used and inequalities in compensation were balanced out by a method involving two readings on the bridge (7).

The platinum coil was made of 0.002 in. wire ($r_0 = 30$ ohms) obtained from Johnson-Matthey, with a specified value for δ (Equation (1)) of 1.49. The lead coil was of 0.003 in. extruded wire ($r_0 = 9$ ohms) obtained from the Baker Chemical Company, and was rated of high purity.

The platinum thermometer was calibrated in ice, steam, liquid oxygen, liquid hydrogen, and liquid helium in the usual way. The last two points proved to be of no interest from the point of view of calibration. In addition, a careful calibration was made between $-183°$ C. and $-210°$ C. by means

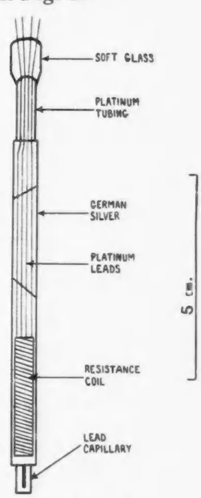

FIG. 2.
Resistance thermometer.

of an oxygen vapor pressure thermometer. Pure gas was obtained for this purpose by heating potassium permanganate in a vacuum.

The values of the resistance ratio $\left(R = \frac{r}{r_0}\right)$ obtained for the various temperatures are given in Table I. Repeated calibrations at some of these points showed no noticeable change over a period of a year.

TABLE I

RESISTANCE RATIO FOR PLATINUM THERMOMETER

Temp., °C.	R	Temp., °C.	R
100.00	1.38976	−191.01	0.21407
0.00	1.00000	−195.73	0.19377
−182.98	0.24891	−210.41	0.13152
−187.76	0.22823	−252.78	0.00957
−188.66	0.22434	−269.0	0.00438

The lead thermometer was calibrated in ice and in liquid hydrogen, and indirectly for the range −180° to −210° C. by comparison with the platinum thermometer during the actual experiments. A slight zero shift was observed after the thermometer had been immersed in liquid air or in liquid hydrogen. The values of R obtained for the oxygen and hydrogen points were 0.29725 at −182.98° C. and 0.03615 at −252.78° C.

Resistance of Platinum. The international Temperature Scale (3) is defined in terms of the platinum resistance thermometer by the equation,

$$t - t_{pt} = \delta\left(\frac{t^2}{100^2} - \frac{t}{100}\right) + \gamma\,(t - 100)\,t^3\ , \tag{1}$$

for $0° > t > -193°$ C. However, it has been found by Heuse and Otto (10), and by Keesom and Dammers (11), that temperatures derived by means of Equation (1) depart from the thermodynamic scale by as much as 0.05° C. at −80° C. and −140° C., and in opposite directions.

Henning (9) has deduced that the resistances of different specimens of platinum should satisfy the relation,

$$\Delta R = a(R - 1) + b(R - 1)^2\ , \tag{2}$$

where R is the resistance ratio r/r_0. Onnes (13) has shown that this relation holds to −215° C. with an accuracy in the derived temperatures of 0.02° C. Below this temperature no satisfactory relation has been found for comparing the resistances of different samples.

Temperatures were calculated directly by means of equation (1) from the resistance measurements of the platinum thermometer, and also by comparison with a detailed standard calibration by means of Equation (2). The standards used were Henning's thermometer No. 29 to −195° C., and Onnes' No. 23′ from −195° to −210° C., the values of R for the standards being taken from

Leiden Comm. Supp. 58. Temperatures calculated by the two methods differed at some points by as much as 0.06° C. This gave rise to differences of as much as 0.1% in the temperature intervals. This is in agreement with the results of Heuse and Otto, and Keesom and Dammers. The method of comparison by Equation (2) was found to give close agreement with the calibration by means of the oxygen vapor pressure thermometer. The latter method was therefore adopted for the calculation of temperatures to −210° C.

Resistance of Lead. It has generally been found that the resistance ratios R, R', of different specimens of lead can be compared by means of the relation,

$$R' = \frac{R - a}{1 - a},$$

which reduces to

$$\Delta R = a(R - 1). \tag{3}$$

Temperatures were first calculated by means of Equation (3), using as a standard a specimen calibrated in detail by Onnes (5) and deriving the constant a from the resistance at the hydrogen point. This method gave only approximate agreement with the calibration points at higher temperatures, and consequently the same Equation (2) that was used for platinum was also adopted for lead, with the constants a and b determined from the hydrogen and oxygen points. A further small correction term was added to allow for a discrepancy of 0.04° at −210° C.

Experimental Measurements

To test the reliability of the calorimeter under widely different experimental conditions some determinations of the specific heat, with the copper specimen above room temperature, were made; *(A)* with the calorimeter in a water bath, and *(B)* in a bath of liquid air. With the large calorimeter the values obtained from *(B)* were 0.15% higher than those from *(A)*. In view of a similar correction in the previous investigation (4) it was thought that, with a liquid air bath, the correction would be proportional to the heat loss from the jacket. The measurements at temperatures above that of liquid air were made with this calorimeter. However, it was feared that at liquid hydrogen temperatures, where the heat capacity of copper is low, the above correction might have a larger value, and therefore the smaller calorimeter was constructed for use in the lower temperature region. Results obtained above room temperature with this new calorimeter for Cases *(A)* and *(B)* agreed within 0.02% of each other and within 0.05% of those obtained with the large calorimeter for Case *(A)*. It was found that, when the differential thermocouples were kept balanced, the temperature of the specimen did not change by more than 0.001° C. in one hour.

Measurements were made with the small calorimeter from 30° to 90° K. and two determinations were made at 93° K. and 133° K. At both these temperatures the results were 0.15% lower than those obtained with the large calorimeter. This showed that the correction factor did not vary as first supposed but was constant to 90° K.

Table II gives the values obtained for the specific heat of copper from 30° to 200° K. Temperatures are given in the Kelvin scale (0° C. = 273.15° K.). The value of C_p is given in joules per gram and in calories per mole (1 $\text{Cal.}_{15°}$ = 4.1835 electrical joules). The values of C_v are calculated from the relation

$$C_p - C_v = 1.60 \times 10^{-5} C_p^2 T .$$

The values of the characteristic temperature, θ, were found from a table of Debye functions given by Beattie (1). A value of 5.9613 calories per mole was used for $3R$.

Since some of the temperature intervals were as great as 8° C., calculations were made to find the difference between the mean specific heat over the range in question and the true specific heat at the mean temperature. For all except the two lowest points the difference proved to be within the limits of experimental error. For the first and second points however the correction amounted to about 1%. This correction was quite accurately determined by a graphical method.

TABLE II

THE SPECIFIC HEAT OF COPPER

Temp., °K.	C_p Joules per gm.	C_p Cal. per mole	$C_p - C_v$ Cal. per mole	C_v Cal. per mole	θ
28.64	0.02280	0.3475	0.0002	0.3473	314.0
35.93	0.04440	0.6745	0.0005	0.6740	310.5
42.58	0.06985	1.047	0.001	1.046	309.1
50.13	0.09855	1.497	0.002	1.495	309.5
59.24	0.1339	2.035	0.004	2.031	310.7
67.21	0.1620	2.462	0.007	2.455	312.7
74.64	0.1870	2.840	0.009	2.831	312.9
87.45	0.2239	3.402	0.016	3.386	313.0
87.88	0.2249	3.417	0.016	3.401	313.2
92.79	0.2365	3.594	0.019	3.575	313.8
93.18	0.2373	3.606	0.019	3.587	314.0
97.41	0.2463	3.743	0.022	3.721	314.9
103.08	0.2579	3.919	0.025	3.894	315.1
108.51	0.2681	4.074	0.029	4.045	315.3
113.73	0.2772	4.212	0.032	4.180	314.9
119.38	0.2861	4.347	0.036	4.311	314.9
125.42	0.2947	4.467	0.040	4.438	314.6
131.29	0.3025	4.597	0.044	4.553	313.9
132.97	0.3043	4.624	0.045	4.579	314.4
137.48	0.3092	4.699	0.049	4.650	314.8
144.24	0.3171	4.818	0.054	4.764	312.9
151.08	0.3236	4.917	0.058	4.859	312.3
157.79	0.3293	5.004	0.063	4.941	312.0
164.71	0.3345	5.083	0.068	5.015	312.0
171.83	0.3399	5.165	0.073	5.092	310.4
178.84	0.3443	5.232	0.078	5.154	309.9
186.33	0.3486	5.297	0.084	5.213	309.6
194.29	0.3528	5.361	0.089	5.272	308.6

Discussion of Results

The accuracy of the results is limited in general by that of the temperature measurements. Above 80° K. it is considered to be within 0.05%. Below this temperature the accuracy is somewhat reduced because of uncertainties in the use of the lead thermometer. An error of more than 0.01° in a 5° interval in this range is considered unlikely.

Fig. 3 gives the values of θ plotted against the temperature. The corresponding values for the specific heat can be found from Beattie's table. Above 90° K. only one point lies off the curve by an amount corresponding to more than 0.05% in C_p. The rise in the curve at 70° K. corresponds to about 0.3% in C_p and is thought to be well within the limits of error. The existence

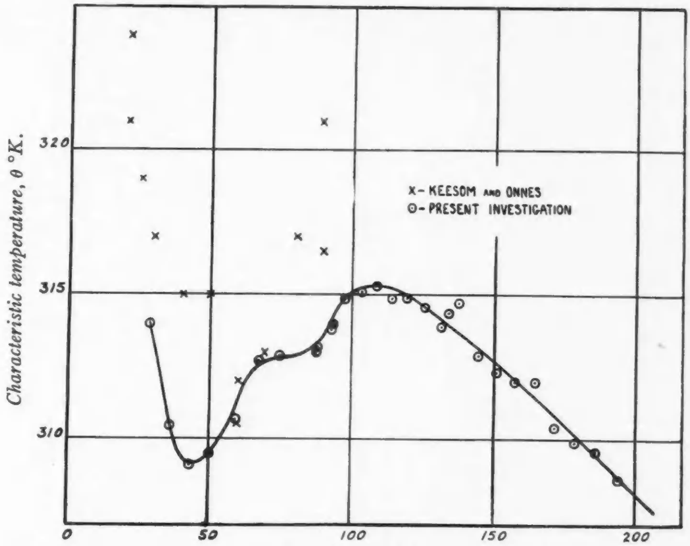

FIG. 3. *Characteristic temperature, θ, for copper.*

of a minimum at about 50° K. is in fair agreement with the work of Keesom and Onnes (12). Their points show a maximum deviation from the present curve of about 3%, which was the accuracy they had claimed. Both show an increase in the value of θ at lower temperatures.

Above 110° K. the curve falls quite rapidly. In the previous work above 200° K. it was found that the specific heat curve could be fitted with a constant value of θ if a term, linear with the temperature, were added to Debye's equation. This equation was found to hold, in the present case, to 100° K. with an accuracy of 1%. Below this temperature however the equation failed, since it amounted roughly to a linear correction in θ.

The results agree with those of Griffiths and Griffiths (8) at room temperature but fall below their curve by 4% at 140° K. The results agree however at this temperature with the results of Eucken and Werth (6).

In the previous work (2, 4) differences of 0.2% were found in the specific heat of specimens of specific gravities 8.8 and 8.9. The specific gravity of the specimen used in the previous work from −78° to 0° C. was 8.90, while that of the specimen used in the present work was 8.91. The two should have specific heats very nearly equal. Later unpublished work has shown

that the values previously obtained (4) are too low by 0.2%. When this correction is made the two agree to within 0.05%, both at room temperature and at 200° K.

Acknowledgments

The author wishes to express his indebtedness to Dr. E. F. Burton for his supervision and interest in this work, and to Mr. J. O. Wilhelm for valuable advice and assistance with the low temperature technique.

References

1. Beattie, J. A. J. Math. Phys. 6 : 1-32. 1926.
2. Bronson, H. L., Chisholm, H. M. and Dockerty, S. M. Can. J. Research, 8 : 282-303. 1933.
3. Burgess, G. K. Bur. Standards J. Research, 1 : 635-640. 1928.
4. Dockerty, S. M. Can. J. Research, 9 : 84-93. 1933.
5. Eucken, A. Handbuch. der Experimentalphysik. VIII, 1 : 64. Akademische Verlagsgesellschaft m.b.H., Leipzig. 1929.
6. Eucken, A. and Werth, H. Z. anorg. Chem. 188 : 152-172. 1930.
7. Glazebrook, Sir Richard. Dictionary of applied physics; Resistance thermometry. Vol. 1. Macmillan and Company, Ltd. London.
8. Griffiths, E. H. and Griffiths, E. Trans. Roy. Soc. London, A, 214 : 319-357. 1914.
9. Henning, F. Ann. Physik, 40 : 635-667. 1913.
10. Heuse, W. and Otto, J. Ann. Physik, 9 : 486-504. 1931.
11. Keesom, W. H. and Dammers, B. G. Physica, 2 : 1080-1090. 1935.
12. Keesom, W. H. and Onnes, H. K. Proc. Acad. Sci. Amsterdam, 18 : 484-493. 1916.
13. Leiden Communication Supplement 58. 1926.